A three-dimensional model of $SL(2,\mathbb{R})$ and the hyperbolic pattern of $SL(2,\mathbb{Z})$

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Abstract

The special linear group $SL(2,\mathbb{R})$, the group of 2×2 real matrices with determinant one, is one of the most important and fundamental mathematical objects not only in mathematics but also in physics. In this paper, we propose a three-dimensional model of $SL(2,\mathbb{R})$ in $\mathbb{R}^3$, which is realized by embedding $SL(2,\mathbb{R})$ into the unit 3-sphere. In this model, the set of symmetric matrices of $SL(2,\mathbb{Z})$ forms a hyperbolic pattern on the unit disk, like the islands floating on the sea named $SL(2,\mathbb{R})$. The structure of this hyperbolic pattern is described in the upper half-plane H. The upper half-plane H also enables us to generate symmetric matrices of $SL(2,\mathbb{R})$ with three circles. Furthermore, the well-known fact $H = SL(2,\mathbb{R})/SO(2)$ is visualized as S^1 fibers of Hopf fibration in the unit 3-sphere. With this three-dimensional model in $\mathbb{R}^3$, we can have a concrete image of $SL(2,\mathbb{R})$ and its noncommutative group structure. This kind of visualization might bring great benefits for the readers who have background not only in mathematics, but also in all areas of science.

1 Introduction

The purpose of this paper is to propose a three-dimensional model of $SL(2,\mathbb{R})$ in $\mathbb{R}^3$. The special linear group $SL(2,\mathbb{R})$, the group of 2×2 real matrices with determinant one, is one of the most important and fundamental mathematical objects not only in mathematics (see, [7, 9, 10]) but also in physics (see, [1, 5]). Nevertheless, it is difficult for us to grasp the whole image of $SL(2,\mathbb{R})$ and its noncommutative group structure. The three-dimensional model of $SL(2,\mathbb{R})$ is realized by embedding $SL(2,\mathbb{R})$ into the unit 3-sphere. By the stereographic projection from the unit 3-sphere into $\mathbb{R}^3$, we can visualize every element in $SL(2,\mathbb{R})$ as a point in $\mathbb{R}^3$. In this three-dimensional model, the set of symmetric matrices of $SL(2,\mathbb{Z})$ forms a hyperbolic pattern on the unit disk as shown in Figure 1. This hyperbolic pattern is regarded as a visualization of the well-known fact $H = SL(2,\mathbb{R})/SO(2)$, where H is the hyperbolic plane and $SO(2)$ is the special orthogonal group in dimension 2.

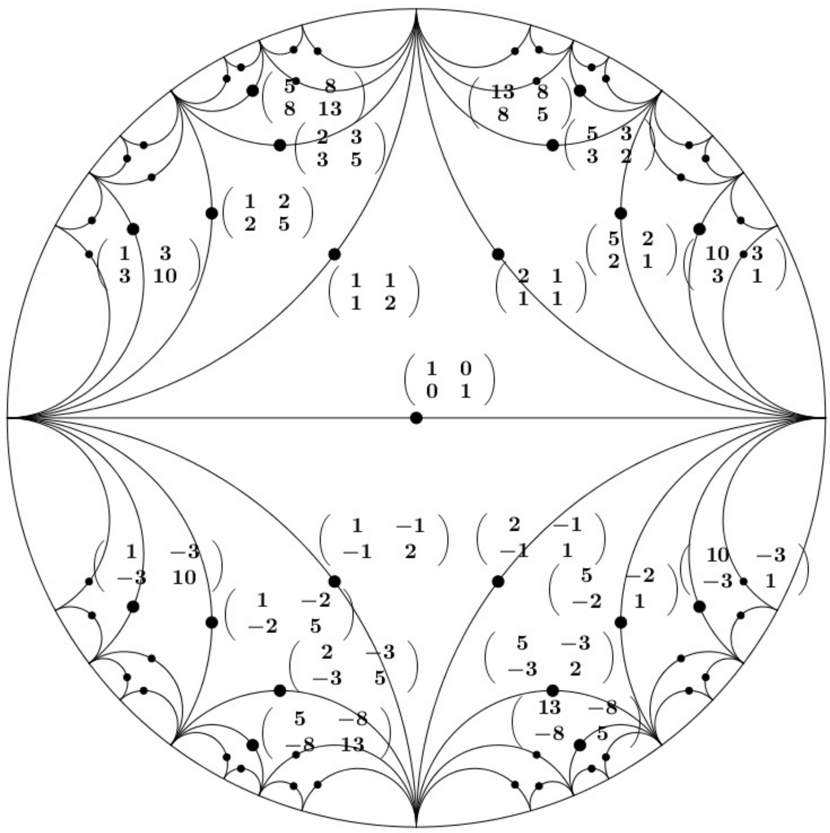

Figure 1: Hyperbolic pattern of $SL(2, \mathbb{Z})$.

In Section 2, we construct the three-dimensional model of $SL(2, \mathbb{R})$ in $\mathbb{R}^3$. In Section 3, we focus on the hyperbolic pattern of the set of symmetric matrices of $SL(2, \mathbb{Z})$. Finally, the well-known fact $H = SL(2, \mathbb{R})/SO(2)$ is visualized as S^1 fibers of Hopf fibration (see, [3] pp.320–323, [4] pp. 298–305) in the model in Section 4.

2 Three-dimensional model of $SL(2, \mathbb{R})$

In this section, we propose a three-dimensional model of $SL(2, \mathbb{R})$. The real special linear group

$$SL(2, \mathbb{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL(2, \mathbb{R}) \;\middle|\; ad - bc = 1 \right\}$$

is embedded into the three-dimensional unit sphere

$$S^3 = \left\{ (u, v) \in \mathbb{C}^2 \;\middle|\; |u|^2 + |v|^2 = 1 \right\}.$$

To see this, let C_0 be a great circle in S^3 defined by

$$C_0 = \left\{ (0, e^{i\theta}) \in S^3 \;\middle|\; \theta \in [0, 2\pi) \right\}.$$

For a point $(u,v)\in S^3\backslash C_0$, the real 2×2 matrix

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\frac{1}{|u|^2}\begin{pmatrix} \mathrm{Re}(u)+|u|\mathrm{Re}(v) & -\mathrm{Im}(u)+|u|\mathrm{Im}(v) \\ \mathrm{Im}(u)+|u|\mathrm{Im}(v) & \mathrm{Re}(u)-|u|\mathrm{Re}(v) \end{pmatrix} \tag{1}$$

is an element of $SL(2,\mathbb{R})$, because the determinant of A is equal to one. The embedding of $SL(2,\mathbb{R})$ into S^3 is given as the inverse map $\pi_0 : SL(2,\mathbb{R})\to S^3\backslash C_0$ determined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mapsto (u,v)=\left(\frac{2}{r^2}\{(a+d)+i(-b+c)\},\frac{1}{r}\{(a-d)+i(b+c)\}\right), \tag{2}$$

where $r=\sqrt{(a+d)^2+(-b+c)^2}$ $(=2/|u|)$.
The stereographic projection (see, [3] p.260, [4] p.74, [6] pp.74–77) of S^3 from the south pole $(u,v)=(-1,0)$ to the three-dimensional Euclidean space $\mathbb{R}^3$ such that

$$(X,Y,Z)=\frac{(\mathrm{Re}(v),\mathrm{Im}(v),\mathrm{Im}(u))}{1+\mathrm{Re}(u)}$$

enables us to visualize almost every element in $SL(2,\mathbb{R})$ as a point in $\mathbb{R}^3$. Only one invisible element $-I_2=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is at infinity, because $-I_2$ corresponds to the south pole $(u,v)=(-1,0)$ in S^3. In this way, we obtain the projection $\Pi_0 : SL(2,\mathbb{R})\to\mathbb{R}^3\cup\{\infty\}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mapsto (X,Y,Z)=\frac{(\mathrm{Re}(v),\mathrm{Im}(v),\mathrm{Im}(u))}{1+\mathrm{Re}(u)}=\frac{(r(a-d),r(b+c),2(-b+c))}{r^2+2(a+d)}.$$

The typical subgroups: diag(e^t,e^{-t}), $SO(1,1)$, and $SO(2)$ are projected into the X,Y,Z-axes respectively (see, [8]):

$$\Pi_0\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right)=\left(\tanh\frac{t}{2},0,0\right),\quad \Pi_0\left(\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\right)=\left(0,\tanh\frac{t}{2},0\right),$$
$$\Pi_0\left(\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\right)=\left(0,0,\tan\frac{t}{2}\right).$$

In the next section, we focus on the set of symmetric matrices of $SL(2,\mathbb{R})$.

3 Hyperbolic pattern of symmetric matrices of $SL(2,\mathbb{Z})$

3.1 Hyperbolic pattern on the Poincaré disk

Let Sym^+ be the set of symmetric matrices with positive trace:

$$Sym^+=\left\{\begin{pmatrix} a & b \\ b & d \end{pmatrix}\in SL(2,\mathbb{R})\,\middle|\, a+d>0\right\}.$$

Then, the value of r in the map Π_0 is equal to $a+d$. The range of the restriction of Π_0 to Sym^+ is the open unit disk in the XY-plane, because $X^2+Y^2=\dfrac{a+d-2}{a+d+2}<1$ and $Z=0$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we obtain the map $\pi_1 : Sym^+ \to D=\{z\in\mathbb{C} \mid |z|^2<1\}$ such that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \mapsto z=\pi_1\left(\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right)=\frac{a-d}{a+d+2}+i\frac{2b}{a+d+2}.$$

Figure 1 shows that the elements of $SL(2,\mathbb{Z})$ form a hyperbolic pattern on the unit disk. The identity matrix $I_2=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is at the origin. For any element A in Sym^+, the sequence of matrices $\{\cdots,A^{-2},A^{-1},A^0(=I_2),A,A^2,\cdots\}$ is arranged in a line. This hyperbolic pattern is precisely described with the upper half-plane model of hyperbolic geometry.

3.2 Description of the hyperbolic pattern in the upper half-plane

Let φ be the transformation from the open unit disk D to the upper half-plane $H=\{w\in\mathbb{C} \mid \mathrm{Im}(w)>0\}$ defined by

$$z\mapsto w=\varphi(z)=i\frac{-z+1}{z+1}.$$

With this transformation φ, we obtain the map $\pi_2=\varphi\circ\pi_1$ from Sym^+ to H such that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \mapsto w=\pi_2\left(\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right)=\frac{b}{a}+i\frac{1}{a}.$$

Figure 2 shows the hyperbolic pattern of $SL(2,\mathbb{Z})$ in H. This pattern is invariant under the next two

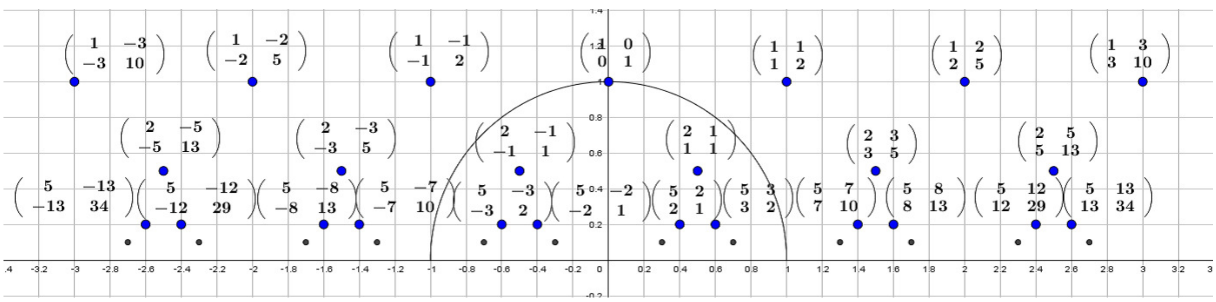

Figure 2: Upper half-plane model of Sym^+.

transformations in H:

$$f(w)=-\frac{1}{w},\quad g(w)=w+1,$$

as shown in the following argument. Here, let us recall that these two transformations f and g are the generators of the modular group $PSL(2,\mathbb{Z})$ (see, [2] pp. 229–230):

$$PSL(2,\mathbb{Z})=< f,g \mid f^2=(fg)^3=g^\infty=id > .$$

The hyperbolic pattern of $SL(2,\mathbb{Z})$ in Sym^+ is generated by the generator of $PSL(2,\mathbb{Z})$ by coincidence.

Theorem 1 *The hyperbolic pattern of $SL(2,\mathbb{Z})$ in H is generated by two transformations*

$$f(w) = -\frac{1}{w}, \quad g(w) = w + 1.$$

Proof. It is easy to check that for any $A \in Sym^+$,

$$\pi_2^{-1} \circ f \circ \pi_2(A) = A^{-1}, \quad \pi_2^{-1} \circ g \circ \pi_2(A) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In particular, if $A \in SL(2,\mathbb{Z})$, then both $\pi_2^{-1} \circ f \circ \pi_2(A)$ and $\pi_2^{-1} \circ g \circ \pi_2(A)$ are in $SL(2,\mathbb{Z})$. Therefore, $SL(2,\mathbb{Z})$ is invariant under f and g.
The rest of the proof is to show that $SL(2,\mathbb{Z})$ is transitive in H; for any $A_0 \in SL(2,\mathbb{Z})$ in Sym^+, we can take $\pi_2(A_0)$ to $i(= \pi_2(I_2))$ by finite composition of f and g. If the value a of A_0 is one, then, $\text{Im}(\pi_2(A_0)) = 1$, hence, $\pi_2(A_0)$ is taken to $i(= \pi_2(I_2))$ by g^n for some $n \in \mathbb{Z}$. Otherwise, $\text{Im}(\pi_2(A_0)) \le \frac{1}{2}$. Note that if $|w| < 1$, then $\text{Im}(f(w)) > \text{Im}(w)$. We can choose $n_0 \in \mathbb{Z}$ such that $|g^{n_0}(\pi_2(A_0))| < 1$. By f, $\text{Im}(f \circ g^{n_0}(\pi_2(A_0))) > \text{Im}(g^{n_0}(\pi_2(A_0)))$, therefore, $\text{Im}(f \circ g^{n_0}(\pi_2(A_0))) > \text{Im}(\pi_2(A_0))$. By repeating these procedures, we can eventually take $\pi_2(A_0)$ to $i(= \pi_2(I_2))$. This completes the proof. ■

In this way, it is natural that the metric on Sym^+ is determined as the hyperbolic metric (see, [2] p.127):

$$ds = \frac{2|dz|}{1 - |z|^2}$$

in the Poincaré unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ as shown in Figure 1.

Corollary 2 *With the hyperbolic metric on Sym^+, the minimal distance among the elements of $SL(2,\mathbb{Z})$ is $2\log\phi(\approx 0.9624)$, where ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$. For each element of $SL(2,\mathbb{Z})$, there are four closest elements of $SL(2,\mathbb{Z})$ which form a rectangle. The angle between two diagonals of the rectangle is $\arccos\frac{3}{5}$.*

Proof. By Theorem 1, it is enough to consider the neighborhood of $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By Figure 1, the four closest elements to I_2 are

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \; A_4 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

On the unit disk D, $\pi_1(A_1) = \dfrac{1+2i}{5}$ and $\pi_1(A_2) = \dfrac{-1+2i}{5}$. Direct calculation yields

$$d(I_2, A_1) = \int_0^{|\pi_1(A_1)|} \frac{2}{1-t^2}\,dt = \left[\log\frac{1+t}{1-t}\right]_0^{\frac{\sqrt{5}}{5}} = \log\frac{(1+\sqrt{5})^2}{4} = 2\log\phi.$$

Since the angle subtended by two diagonals A_1A_3 and A_2A_4 is equal to the angle $\angle A_1I_2A_2$ on the complex plane, it follows that

$$\cos\angle A_1I_2A_2 = \frac{-1+4}{5} = \frac{3}{5}.$$

This completes the proof. ■

3.3 Construction of (a, b, d)-triple

At the end of this section, let us introduce an interesting method to make three numbers a, b, and d satisfying $ad - b^2 = 1$ without calculation but with construction of three circles in the upper half-plane. Let ψ be the transformation from D to the upper half-plane $H = \{w \in \mathbb{C} \mid \text{Im}(w) > 0\}$ defined by

$$z \mapsto w = \psi(z) = \frac{z+i}{iz+1}.$$

With this transformation ψ, we obtain the map $\pi_3 = \psi \circ \pi_1$ from Sym^+ to H such that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \mapsto w = \pi_3\left(\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right) = \frac{a-d}{a+d-2b} + i\frac{2}{a+d-2b}.$$

Figure 3 shows the hyperbolic pattern of $SL(2, \mathbb{Z})$ in H.

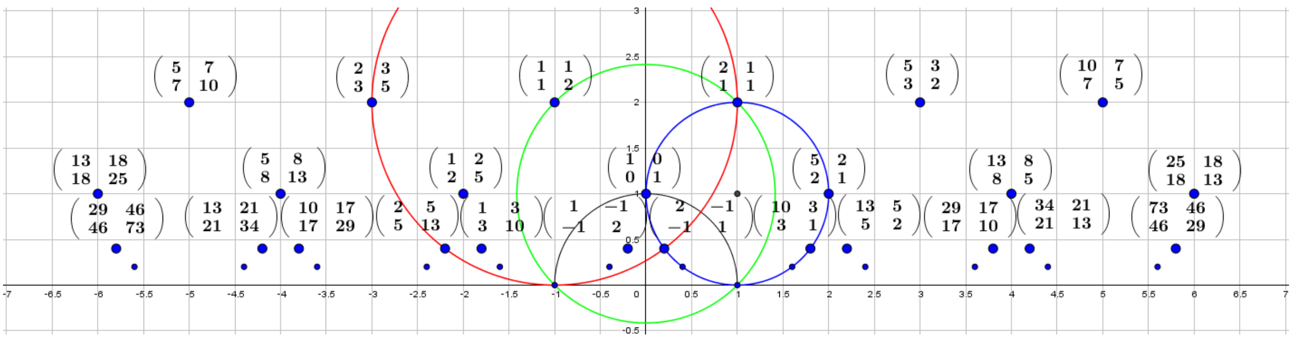

Figure 3: Another upper half-plane model of Sym^+.

In this upper half-plane model, since $ad - b^2 = 1$,

$$|w|^2 = \frac{a+d+2b}{a+d-2b}.$$

By using the equation above, it is easy to check the following equations:

$$|w - (-1 + ia)| = a, \quad |w - (1 + id)| = d, \quad |w - ib| = 1 + b^2. \tag{3}$$

Equations (3) imply that for any point in this upper half-plane model, we can detect the corresponding element of Sym^+. In other words, we can make any (a, b, d)-triple satisfying $ad - b^2 = 1$ by drawing three circles as follows;

'Construction of (a, b, d)-triple which holds $ad - b^2 = 1$ by three circles'

1. Take any point P on the upper half-plane.
2. Draw circle C_1 tangent to the x-axis at $(-1, 0)$ and passing through P.
3. Draw circle C_2 tangent to the x-axis at $(+1, 0)$ and passing through P.

4. Draw circle C_3 passing through $(-1,0)$, $(+1,0)$, and P.

5. Let a, d, b be the y-coordinate of the center of three circles C_1, C_2, C_3, respectively. Then, the (a,b,d)-triple satisfies $ad - b^2 = 1$.

For example, if $P = (1,2)$, then the three circles are given by

$$C_1 : (x+1)^2 + (y-2)^2 = 2^2,\ C_2 : (x-1)^2 + (y-1)^2 = 1^2,\ C_3 : x^2 + (y-1)^2 = 2.$$

Hence, $P = (1,2) \in H$ corresponds to $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in Sym^+$ as shown in Figure 3.

4 Hopf fibrations of $SL(2,\mathbb{R})$

At the end of this paper, let us go back to the map π_0 and the three-dimensional model in Section 2. For $K(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$, let us consider the left and right translations in $SL(2,\mathbb{R})$ defined by

$$K_L(\theta)(A) = K(\theta)A,\quad K_R(\theta)(A) = AK(\theta)$$

for $A \in SL(2,\mathbb{R})$. Then, by Equations (1) and (2) in Section 2,

$$\pi_0 \circ K_L(\theta) \circ \pi_0^{-1}(u,v) = (e^{i\theta}u, e^{i\theta}v),\quad \pi_0 \circ K_R(\theta) \circ \pi_0^{-1}(u,v) = (e^{i\theta}u, e^{-i\theta}v).$$

Therefore, the orbits under these translations are great circles in S^3. The stereographic projection maps circles to circles, and hence, these two orbits are two different circles on the same torus as shown in Figures 4 and 5. These figures illustrate clearly the noncommutativity of the group structure

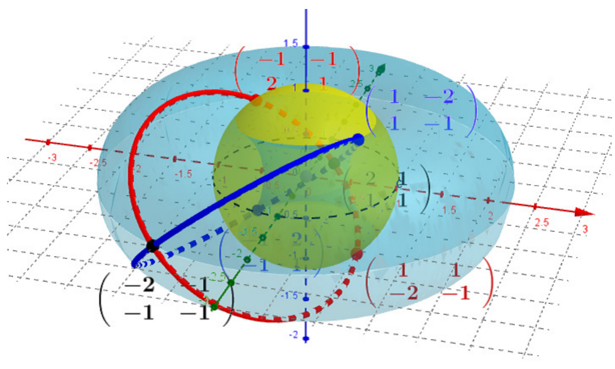

Figure 4: Left and right translations.

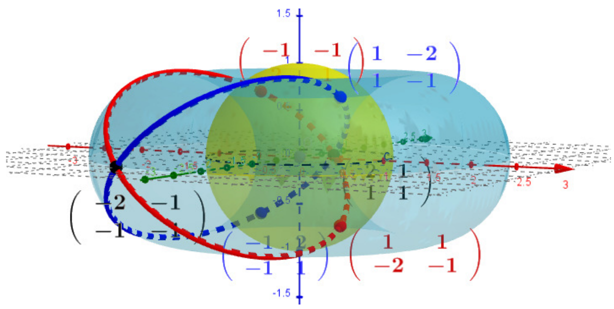

Figure 5: Side view of Figure 4.

of $SL(2,\mathbb{R})$. Furthermore, Figures 4 and 5 show that $SL(2,\mathbb{R})$ is the product of Sym^+ and $SO(2)$, because every S^1 fiber of the left translation meets at one point in Sym^+. Since Sym^+ may be regarded as the Poincaré disk with hyperbolic metric which isometric to the upper half-plane H, the well-known fact $H = SL(2,\mathbb{R})/SO(2)$ can now be visualized in the three-dimensional model.

5 Closing remarks

In this paper, we have proposed a three-dimensional model of $SL(2,\mathbb{R})$. The set of symmetric matrices corresponds to the hyperbolic plane H, which entails the well-known fact:

$$H = SL(2,\mathbb{R})/SO(2). \tag{4}$$

In general, Equation (4) is derived by the following algebraic approach: $SL(2,\mathbb{R})$ acts on the homogeneous space H as the Möbius transformation, and the point stabilizer of $i \in H$ is $SO(2)$. In this sense, the three-dimensional model gives us another approach for Equation (4). Noncommutativity of the group structure of $SL(2,\mathbb{R})$ is also visualized. In this way, the three-dimensional model is useful for understanding the group $SL(2,\mathbb{R})$. The complete visualization of the group structure forms part of our future work.

References

[1] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Annals of Mathematics Vol. 48, No. 3 (1947), pp. 568–640.

[2] A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, 1983.

[3] M. Berger, *Geometry I*, Springer-Verlag, 1987.

[4] M. Berger, *Geometry II*, Springer-Verlag, 1987.

[5] Harish-Chandra, *Plancherel formula for the 2 × 2 real unimodular group*, Proc. Natl. Acad. Sci. U.S.A., Vol. 38, No. 4 (1952), pp. 337–342.

[6] G. Jennings, *Modern geometry with applications*, Springer-Verlag, 1994.

[7] S. Lang, $SL_2(\mathbf{R})$, Springer-Verlag, 1985.

[8] Y. Maeda, *Active Learning with Dynamic Geometry Software*, ICCSA 2017, Part IV, LNCS **10407** (2017), pp. 228–239.

[9] E. Molnár, *The projective interpretation of the eight 3-dimensional homogenuouse geometries*, Beiträge, Algebra Geom. Vol. 38 (1997), pp. 261–288.

[10] W. P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. Vol. 6, No. 3 (1982), pp. 357–381.